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The geometrically invariant form of evolution equations

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Abstract
We study the evolution of geometric invariants for equations such as the Davey–Stewartson and Novikov–Veselov equations.

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1. Introduction

A large class of evolution equations in two space, (x, y), and one time, t, dimension arise as compatibility conditions on the (x, y, t)-dependent coefficients in linear differential operators (Lax operators). Examples include the Kadomtsev–Petviashvili, Davey–Stewartson and Novikov–Veselov (NV) equations. Thus the linear operators

$$\mathbb{L} = \begin{pmatrix} \partial_x & q \\ -1 & \partial_y \end{pmatrix}$$

and

$$\mathbb{M} = \begin{pmatrix} \partial_Y^3 - 3U_{yY}\partial_Y + 3U_{yyY} & 3q_Y\partial_Y - 3q_{yY} - 3(U_{xx} - U_{yy})q \\ 3(U_{xx} - U_{yy}) & \partial_Y^3 + 3U_{xY}\partial_Y - 3q_Y \end{pmatrix}$$

where $U_{xy} = q$ and $\partial_Y = \partial_x - \partial_y$, are associated in this way with the NV-equation

$$q_t = q_{xxx} - q_{yyy} + 3(qU_{xx})_x - 3(qU_{yy})_y.$$

Here the subscripts denote derivatives.

The form the equation takes is, of course, dependent upon the form of the Lax operators. This form corresponds to a specific choice of 'gauge' because neither the linearity nor compatibility of $\mathbb{L}$ and $\mathbb{M}$ is compromised by the transformation $\mathbb{L} \to \mathbb{L}^g = g^{-1}\mathbb{L}g$, where g is a 2×2 matrix of arbitrary functions. (In order to preserve the leading order, differential part of $\mathbb{L}$ it is necessary that g be diagonal.) On the other hand the forms of $\mathbb{L}$ and $\mathbb{M}$ chosen above are not of the most obviously general kind. In the case of $\mathbb{L}$ the general form for a hyperbolic linear operator can be taken to be

$$\mathbb{L} = \begin{pmatrix} \partial_x + h_{11} & h_{12} \\ h_{21} & \partial_y + h_{22} \end{pmatrix} \tag{1}$$

the h_{ij} being real functions of x, y and t. Starting with such a general form one might construct the associated $\mathbb{M}$ and write down evolution equations for the h_{ij} but this would again mean ignoring the role of gauge transformations and obscure the geometrical meaning of these evolution equations. Instead one should construct evolution equations for the gauge invariant quantities [3, 4]

$$(12) = h_{12}h_{21} \tag{2}$$

$$[12] = h_{11x} - h_{22y} + \frac{1}{2}\log\left(\frac{h_{12}}{h_{21}}\right)_{xy}. \tag{3}$$

The notation for these quantities is borrowed from tensor conventions and is intended to convey the symmetry (respectively antisymmetry) of these quantities under permutations of the labels 1 and 2.

These independent quantities label the equivalence classes under gauge transformations because, in fact, $\mathbb{L}$ and $\mathbb{L}'$ are gauge equivalent *if and only if* their respective invariants coincide: $(12) = (12)'$, $[12] = [12]'$. In the case of the Novikov–Veselov equation above, we see that $(12) = q$ and $[12] = \frac{1}{2}\log q_{xy}$ representing an invariant differential constraint,

$$2[12](12)^2 = (12)(12)_{xy} - (12)_x(12)_y. \tag{4}$$

The evolution equations for the gauge invariants (12) and [12] with any attendant differential constraints on the invariants, we will call, in this paper, the *geometrically invariant form* of the evolution equation.

One might ask what such a form serves. Part of the answer has to be that in the transformation theory of such equations (Darboux, Moutard, Bäcklund, etc) it is the invariants which most simply express the transformations and, indeed, from which they are most easily derived. In addition, it is arguable that evolution equations should be classified according to a set of invariant forms. In this paper we will calculate invariants for linear 2×2 matrix operators of orders 2 and 3 (AKNS Lax operators) and give results on the associated evolution equations in geometrically invariant form. The calculations underlying these forms are by no means short and we do not propose to present them since they are in principle routine. We shall also discuss the compatibility of these flows with Laplace transformations.

The study of these invariants, their generalizations and transformations, is both classical [4] and modern. They occur in geometry [1, 8, 9, 12] and the theory of hydrodynamic Hamiltonian systems [7]. For a recent review of their application to the classification of integrable equations see [13]. Within the context of integrable systems it is entirely natural to consider the time evolution of these invariants.

2. Invariant forms of second-order equations

The calculation of the above invariants (12) and [12] for the first-order Lax operator (1) is classical and simple. Rather than calculate invariants for quite general higher order operators directly we will restrict attention to those of interest to evolution equations, namely those which commute with (1). Let $\partial_t - \mathbb{M}$ be the general form of the commuting matrix operator

$$\mathbb{L}_t + [\mathbb{L}, \mathbb{M}] = 0$$

of degree two in ∂_x and ∂_y. This system will correspond to the second member of the $(2+1)$-dimensional AKNS hierarchy. As is customary for equations in Sato-like hierarchies [10] we take the operator parts of $\mathbb{M}$ to be expressible solely in terms of $\partial_Y = \partial_x - \partial_y$ so that

$$\mathbb{M} = \begin{pmatrix} \partial_Y^2 + f_{11}\partial_Y + k_{11} & 2h_{12}\partial_Y + k_{12} \\ 2h_{21}\partial_Y + k_{21} & -\partial_Y^2 + f_{22}\partial_Y + k_{22} \end{pmatrix} \tag{5}$$

and the resulting evolution equations are

$$h_{12t} = h_{12YY} + 2h_{12}h_{22Y} + h_{12Y}f_{11} + h_{12}(k_{11} - k_{22}) + (h_{22} - h_{11})k_{12} - k_{12x} \tag{6}$$

$$h_{21t} = -h_{21YY} + 2h_{21}h_{11Y} + h_{21Y}f_{22} + h_{21}(k_{22} - k_{11}) + (h_{11} - h_{22})k_{21} - k_{21y} \tag{7}$$

$$h_{11t} = h_{11YY} + 2h_{12}h_{21Y} + h_{11Y}f_{11} - h_{12}k_{21} + h_{21}k_{12} - k_{11x} \tag{8}$$

$$h_{22t} = -h_{22YY} + 2h_{21}h_{12Y} + h_{22Y}f_{22} + h_{12}k_{21} - h_{21}k_{12} - k_{22y}. \tag{9}$$

Gauge transformations, $\mathbb{M} \to \mathbb{M}^g = g^{-1}\mathbb{M}g$, preserving the leading order differential operator part of $\mathbb{M}$ are, as before, 2×2 matrices of the form

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}.$$

Gauge invariants are found to be, in addition to (12) and [12], those already known for $\mathbb{L}$,

$$[12]^f = f_{11} + f_{22} + \ln\left(\frac{h_{12}}{h_{21}}\right)_Y \tag{10}$$

$$[12]^g = h_{12}k_{21} - h_{21}k_{12} + (12)\ln\left(\frac{h_{12}}{h_{21}}\right)_Y \tag{11}$$

$$[11] = k_{11} - \tfrac{1}{4}f_{11}^2 - \tfrac{1}{2}f_{11Y} \tag{12}$$

$$[22] = k_{22} + \tfrac{1}{4}f_{22}^2 - \tfrac{1}{2}f_{22Y}. \tag{13}$$

In fact $[12]^g = (12)_x + (12)_y$. The geometrically invariant form of the evolution equations in this case is

$$(12)_t = 2\left\{(12)\int[12]\,\mathrm{d}y\right\}_x + 2\left\{(12)\int[12]\,\mathrm{d}x\right\}_y \tag{14}$$

$$[12]_t = -2(12)_{xx} - 2(12)_{yy} + \left\{\left(\int[12]\,\mathrm{d}x\right)^2 + \left(\int[12]\,\mathrm{d}y\right)^2 + \frac{1}{2}\ln(12)_{xx} + \frac{1}{2}\ln(12)_{yy} + \frac{1}{4}\ln(12)_x^2 + \frac{1}{4}\ln(12)_y^2\right\}_{xy}. \tag{15}$$

Obtaining these equations is by no means a simple calculation although in principle one need only differentiate (12) and [12] with respect to t and use the equations (6)–(9). The above is no more general than the most general equations considered in the literature [11]. The $(2+1)$-dimensional AKNS system, for which the choices $h_{11} = h_{22} = 0$, $h_{12} = p$ and $h_{21} = q$ are conventionally made, has invariants $(12) = pq$ and $[12] = \frac{1}{2}\log\left(\frac{p}{q}\right)_{xy}$. Each form is a family depending upon two arbitrary functions: p and q, or (12) and [12]. Under these choices equations (14) and (15) become

$$p_t q + p q_t = q\Delta p - p\Delta q \tag{16}$$

$$p_t q - p q_t = q\Delta p + p\Delta q + 2pqV \tag{17}$$

where Δ is the two-dimensional Laplacian and $V_{xy} = -2\Delta(pq)$. These are indeed the AKNS second-order flow. On the surface the invariant form has little to recommend it, being far more complex than the conventional form. It takes on a slightly less forbidding aspect if we define new invariants:

$$\Theta = \tfrac{1}{2}\log(12) \tag{18}$$
$$\Psi = \partial_x^{-1}\partial_y^{-1}[12] \tag{19}$$

yielding

$$\Theta_t = \Delta\Psi + 2\nabla\Theta\cdot\nabla\Psi \tag{20}$$
$$\Psi_t = \Delta\Theta + \nabla\Theta\cdot\nabla\Theta + \nabla\Psi\cdot\nabla\Psi + V \tag{21}$$
$$V_{xy} = -4\mathrm{e}^{2\Theta}(\Delta\Theta + 2\nabla\Theta\cdot\nabla\Theta). \tag{22}$$

Here ∇ is the gradient operator and $(\cdot)$ denotes the scalar product in two dimensions.

The reduction to the Davey–Stewartson equation $p = q^*$ corresponds to the invariants (12) and [12] being real and purely imaginary, respectively, together with the restriction to pure imaginary time: $\partial_t = \mathrm{i}\partial_\tau$.

The theory may be further developed for matrix Lax operators where the h_{12} and h_{21} are $n\times m$ and $m\times n$ rectangular and the h_{11} and h_{22} are $n\times n$ and $m\times m$ square matrices. Then (12) and [12] are replaced by matrix covariants. This will be discussed in a further publication.

3. Laplace maps for second-order flows

The Laplace transformation is a map between Lax operators $\mathbb{L}$ and $\mathbb{L}^{\sigma_1}$ and $\mathbb{L}^{\sigma_2}$ satisfying

$$\mathbb{L}\mathbb{D}_x = \mathbb{D}_x^{\sigma_1}\mathbb{L}^{\sigma_1} \tag{23}$$
$$\mathbb{L}\mathbb{D}_y = \mathbb{D}_y^{\sigma_2}\mathbb{L}^{\sigma_2} \tag{24}$$

where $\mathbb{D}_x$, $\mathbb{D}_x^{\sigma_1}$ and $\mathbb{D}_y$, $\mathbb{D}_y^{\sigma_2}$ are distinct matrix operators with differential parts $I\partial_x$ and $I\partial_y$ respectively, I being the unit 2×2 matrix. The notation $\mathbb{L}^{\sigma_1}$, etc denotes the image of the operator $\mathbb{L}$ under the map σ_1, etc defined by (23), (24).

These relations lead to relations between the invariants for the Lax operators of the following form:

$$(12)^{\sigma_1} - (12) = -[12] - \tfrac{1}{2}\ln(12)_{xy} \tag{25}$$
$$[12]^{\sigma_1} - [12] = \tfrac{1}{2}\ln((12)(12)^{\sigma_1})_{xy} \tag{26}$$
$$(12)^{\sigma_2} - (12) = [12] - \tfrac{1}{2}\ln(12)_{xy} \tag{27}$$
$$[12]^{\sigma_2} - [12] = -\tfrac{1}{2}\ln((12)(12)^{\sigma_2})_{xy}. \tag{28}$$

Now the question arises: if (12) and [12] satisfy the invariant form of the second-order evolution equation, is it the case that $(12)^{\sigma_1}$ and $[12]^{\sigma_1}$, etc also satisfy them? Differentiating equation (23) with respect to t and using $\mathbb{L}_t + [\mathbb{L},\mathbb{M}] = 0$ and $\mathbb{L}_t^{\sigma_1} + [\mathbb{L}^{\sigma_1},\mathbb{M}^{\sigma_1}] = 0$ one obtains

$$\mathbb{L}\big(\mathbb{D}_t + \mathbb{D}\mathbb{M}^{\sigma_1} - \mathbb{M}\mathbb{D}\big) = \big(\mathbb{D}_t^{\sigma_1} + \mathbb{D}^{\sigma_1}\mathbb{M}^{\sigma_1} - \mathbb{M}\mathbb{D}^{\sigma_1}\big)\,\mathbb{L}^{\sigma_1}. \tag{29}$$

This relation can be shown, with a lot of work, to be consistent with the relations (25). The following diagram may then be said to commute:

$$\begin{array}{ccc}
\mathbb{L} & \xrightarrow{\ \sigma_1\ } & \mathbb{L}_1^{\sigma} \\
\downarrow & & \downarrow \\
\tilde{\mathbb{L}} & \xrightarrow[\ \tilde{\sigma}_1\ ]{} & (\tilde{\mathbb{L}})^{\tilde{\sigma}_1} = \widetilde{(\mathbb{L}^{\sigma_1})}.
\end{array}$$

Equivalently

$$(\tilde{\mathbb{L}})^{\tilde{\sigma}_1} = \widetilde{(\mathbb{L}^{\sigma_1})}. \tag{30}$$

Under the assumption that the time evolution map $\mathbb{L} \to \tilde{\mathbb{L}}$ is well-defined, this establishes the desirable property of the map σ_1, that it maps the class of solutions of (14) to itself. Similar considerations hold for σ_2.

A further consideration applies to Toda lattice systems

$$-(12)_{n+1} + 2(12)_n - (12)_{n-1} = \ln(12)_{nxy} \tag{31}$$

which are simply the expression of the three term recurrence relationships between sequences of invariants: $(12)_n = (12)^{\sigma_1^n}$, etc. This lattice is integrable in the sense that σ_1 and σ_2 commute in their action on the invariants which can be thought of as a flatness or Lax condition in the x, y space. The commutation conditions (29) now strongly argue for the integrability of the family of t-dependent lattices.

4. Invariant forms of third-order equations and their Laplace maps

Repeating the above for third-order flows leads to the complicated equations:

$$\begin{aligned}(12)_t = (12)_{xxx} - (12)_{yyy} &+ \left((12)\left\{-3\int(12)_x\,\mathrm{d}y + 3\left(\int[12]\,\mathrm{d}y\right)^2 - \frac{3}{4}\log(12)_x^2\right\}\right)_x \\ &+ \left((12)\left\{3\int(12)_y\,\mathrm{d}x - 3\left(\int[12]\,\mathrm{d}x\right)^2 + \frac{3}{4}\log(12)_y^2\right\}\right)_y \end{aligned} \tag{32}$$

$$\begin{aligned}[12]_t = [12]_{xxx} - [12]_{yyy} &- 3\left\{[12]\int(12)_x\,\mathrm{d}y\right\}_x + 3\left\{[12]\int(12)_y\,\mathrm{d}x\right\}_y \\ &- 3\left\{b(12)\right\}_{xx} + 3\left\{c(12)\right\}_{yy} - 3\left\{b(12)_x\right\}_x + 3\left\{c(12)_y\right\}_y \\ &+ \left\{b^3 - c^3 + \frac{3}{2}(ba_x)_x - \frac{3}{2}(ca_y)_y + \frac{3}{4}ba_x^2 - \frac{3}{4}ca_y^2\right\}_{xy}\end{aligned} \tag{33}$$

where $a = \log(12)$, $b = \int[12]\,\mathrm{d}y$ and $c = \int[12]\,\mathrm{d}x$.

Let us rewrite the relations (18) and (19) in the following forms:

$$(12) = \mathrm{e}^{2\Theta}$$
$$[12] = \Psi_{xy}$$

and define a new potential U by

$$(12) = U_{xy} \tag{34}$$

The equations (32) and (33) become, respectively,

$$\Theta_t = \left(\Theta_{xx} + \Theta_x^2 + \tfrac{1}{2}A\right)_x - \left(\Theta_{yy} + \Theta_y^2 + \tfrac{1}{2}B\right)_y + A\Theta_x - B\Theta_y \tag{35}$$

$$\begin{aligned}\Psi_t = (\Psi_{xx} + 3\Theta_x\Psi_x)_x - (\Psi_{yy} + 3\Theta_y\Psi_y)_y + \Psi_x^3 + 3\Psi_x\left(\Theta_x^2 - U_{xx}\right) \\ - \Psi_y^3 - 3\Psi_y\left(\Theta_y^2 - U_{yy}\right) - 3W\end{aligned} \tag{36}$$

where

$$A = \Theta_x^2 + 3\left(\Psi_x^2 - U_{xx}\right)$$
$$B = \Theta_y^2 + 3\left(\Psi_y^2 - U_{yy}\right)$$
$$W_{xy} = (\Psi_x U_{xy})_{xx} - (\Psi_y U_{xy})_{yy}.$$

The geometric constraint (4):

$$C = 2[12](12)^2 - (12)(12)_{xy} + (12)_x(12)_y = 0$$

is respected by the flow in the sense that C_t vanishes when C vanishes, leading to the Novikov–Veselov equation. To see this note that, using $(12) = e^{2\Theta}$ and $[12] = \Psi_{xy}$,

$$C = 2e^{4\Theta}(\Psi - \Theta)_{xy}.$$

So let $\alpha = \Psi - \Theta$ then

$$\alpha_t = \left(\alpha_{xx} - \tfrac{3}{2}\alpha_x^2 + \tfrac{3}{2}U_{xx}\right)_x - \left(\alpha_{yy} - \tfrac{3}{2}\alpha_y^2 + \tfrac{3}{2}U_{yy}\right)_y + \left(\alpha_x^2 - 3U_{xx}\right)\alpha_x - \left(\alpha_y^2 - U_{yy}\right)\alpha_y - 3W.$$

But it is easy to show that the terms not involving the symbol α in this equation satisfy

$$(U_{xxx} - U_{yyy} - 2W)_{xy} = 2\left(\alpha_x e^{2\Theta}\right)_{xx} - 2\left(\alpha_y e^{2\Theta}\right)_{yy}.$$

For $C = 0$ the two members of (32)–(33) reduce to single equation which can then be written solely in terms of the one invariant (12):

$$(12)_t = (12)_{xxx} - (12)_{yyy} - 3\left((12)\int (12)_x\,dy\right)_x + 3\left((12)\int (12)_y\,dx\right)_y. \tag{37}$$

The other familiar reduction is to the modified Novikov–Veselov equation: $h_{11} = h_{22} = 0$, $h_{12} = h_{21} = q$. This is simply the geometric constraint $[12] = 0$. Equation (32) becomes identically zero and the first yields

$$(12)_t = (12)_{xxx} - (12)_{yyy} - \left((12)\left\{3\int (12)_x\,dy + \frac{3}{4}\log(12)_x^2\right\}\right)_x + \left((12)\left\{3\int (12)_y\,dx + \frac{3}{4}\log(12)_y^2\right\}\right)_y. \tag{38}$$

As for the second-order flows the compatibility of these third-order flows with the Laplace transformations is a matter of (an even longer) algebraic verification.

5. Conclusions and prospects

We have presented the lowest members of the (2 + 1)-dimensional AKNS hierarchy, written in terms of the gauge invariants of their basic linear problem. In particular this allows a *geometric* interpretation of reductions, the constraints themselves being relations between gauge invariant objects.

Further, although we have not given the details of the calculations involved, we have also shown that these flows are compatible with Laplace transformations which act naturally on the invariants. We stress that this is the most important aspect of this study firstly because it establishes the foundations of an algebraic solution procedure which is distinct from the usual binary Darboux and Moutard maps and secondly, because it argues for the integrability of time evolutions of Toda-like lattices. In respect of the first aspect it should be noted that Laplace maps *do not* commute with reductions: σ_1 and σ_2 will generally take solutions of, for example, the Novikov–Veselov equation, outside the constraint surface, $C = 0$. The transformed solutions will satisfy the full (2 + 1)-dimensional AKNS system. Laplace maps are not *auto*-Bäcklund transformations of reductions.

The complexity of the evolution equations written in these terms lends a quixotic flavour to the enterprise and it is slightly puzzling that what is quite natural from one point of view appears not to be so from the other. What is needed, if further progress and clarification are to be made, is a cleaner and leaner machine for dealing with these flows. One possibility is the Hamiltonian formulation [2, 5, 6].

References

[1] Anderson I M and Juras M 1997 *Duke Math. J.* **89** 351–75
[2] Athorne C and Dorfman I Y 1993 *J. Math. Phys.* **34** 3507–17
[3] Athorne C 1995 *Phys. Lett.* A **206** 162–6
[4] Darboux G 1897 *Leçons sur la théorie genérale des surfaces* (Paris: Gauthier-Villars)
[5] Dorfman I Y and Athorne C 1993 *Phys. Lett.* A **182** 369–72
[6] Dorfman I Y and Fokas A S 1992 *J. Math. Phys.* **33** 2504
[7] Ferapontov E V 1997 *Theor. Math. Phys.* **110** 68–77
[8] Hesheng Hu 2001 *Lett. Math. Phys.* **57** 19–32
[9] Kamran N and Tenenblat K 2001 *Algebraic Methods in Physics* vol 105–120 (*CRM Series Math. Phys.*) (New York: Springer)
[10] Miwa T, Jimbo M and Date E 2000 *Solitons* (Cambridge: Cambridge University Press)
[11] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)
[12] Schief W K 2000 *J. Math. Phys.* **41** 6566–99
[13] Zhiber A V and Sokolov V V 2001 *Usp. Mat. Nauk* **56** 63–106